Enumerative Combinatorics

Generating Functions

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Basic Ideas

Sequence of finite sets $S_0, S_1, S_2, \dots$
Want to determine or describe $f(i) = \#S_i$

- Closed form formulas
- Open form formulas
- Recurrences
- Estimates
- Generating functions

Generating Functions

$$f(i) = \#S_i$$

Definition

An ordinary generating function is a formal power series with complex coefficients

$$\sum_{n\geq 0} f(n)x^n = \sum_{n\geq 0}\sum_{a\in S_n} x^n \quad \text{or} \quad = \sum_{n\geq 0}\sum_{a\in S_n} w(a)x^n$$

Generating functions of the form $\sum_{n\geq 0} f(n)\frac{x^n}{B(n)}$ may be used.

Examples

Example

Say $\#S_i = 1$ for all i. Then the generating function is

$$1 + x + x^2 + \ \dots \ = \frac{1}{1-x}$$

- In the ring of formal power series over $\mathbb{C}$, there is only one power series that is the inverse of $(1-x)$.

Example

A partition of n is a weakly decreasing sequence of positive integers $\lambda_1, \lambda_2, \dots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_k = n$. Let $p(n)$ be the number of partitions of n. Then

$$\sum_{n \geq 0} p(n)x^n = (1 + x + x^2 + \ \dots\ )(1 + x^2 + x^4 + \ \dots\ ) \dots \ = \prod_{n \geq 1} \frac{1}{1 - x^n}$$

Rational Generating Functions

Definition

A rational generating function has the form

$$\sum_{n \geq 0} f(n)x^n = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials with complex coefficients.

Example

The generating function from the previous slide

$$1 + x + x^2 + ... = \frac{1}{1 - x}$$

is a rational generating function.

When do we have a rational generating function?

Theorem

The following conditions are equivalent:

- $\sum_{n\geq 0} f(n)x^n$ *is a rational generating function* $P(x)/Q(x)$, $Q(x) = 1 + b_1 x + \ldots + b_d x^d$.
- *For fixed* b_i *and* d *and sufficiently large* n, $f(n) + b_1 f(n-1) + \ldots + b_d f(n-d) = 0$.

Proof.

$$Q(x)\sum_{n\geq 0} f(n)x^n = \sum_{n\geq 0}(f(n) + b_1 f(n-1) + \ldots + b_d f(n-d))x^n = P(x)$$

$\square$

Fibonacci Numbers

Theorem

- *$\sum_{n\geq 0} f(n)x^n$ is a rational generating function $P(x)/Q(x)$,* $Q(x) = 1 + b_1x + ... + b_dx^d$.
- *For fixed b_i and d and sufficiently large n,* $f(n) + b_1f(n-1) + ... + b_df(n-d) = 0$.

Example

$f(0) = 0$, $f(1) = 1$, $f(n) - f(n-1) - f(n-2) = 0$ for $n \geq 2$

$\sum_{n\geq 0} f(n)x^n = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 ...$

$$Q(x) = 1 - x - x^2 \text{ and } Q(x)\sum_{n\geq 0} f(n)x^n = x$$

$$\sum_{n\geq 0} f(n)x^n = \frac{x}{1-x-x^2}$$

Evaluation of Some Rational Generating Functions

Lemma

$$(1+x)^j = \sum_{n\geq 0} \frac{j(j-1)\ldots(j-n+1)}{n!}\, x^n$$

for all $j \in \mathbb{C}$.

Lemma

$$\frac{\beta}{(1-\gamma x)^j} = \sum_{n\geq 0} \left(\frac{\beta(n+j-1)(n+j-2)\ldots(n+1)}{(j-1)!}\, \gamma^n \right) x^n$$

for all positive integers j.

Fundamental Property

Theorem

Let $S(x)$ and $Q(x)$ be polynomials over $\mathbb{C}$. Then

$$\frac{S(x)}{Q(x)} = R(x) + \frac{P(x)}{Q(x)}$$

where the degree of $P(x)$ be less than the degree of $Q(x)$.

Theorem

Let the degree of $P(x)$ be less than the degree of $Q(x)$. Then

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{k} \sum_{j=1}^{d_i} \frac{\beta_{ij}}{(1-\gamma_i x)^j}$$

where $Q(x) = \prod_{i=1}^{k}(1-\gamma_i x)^{d_i}$

Fundamental Property

$Q(x) = \prod_{i=1}^{k}(1-\gamma_i x)^{d_i}$ has degree n

Proof.

Letting $P(x)$ vary, we can consider the vector spaces over $\mathbb{C}$

$$V_1 = \left\{ f : \mathbb{N} \to \mathbb{C} \text{ such that } \sum_{n \geq 0} f(n)x^n = \frac{P(x)}{Q(x)},\ deg\ P(x) < n \right\}$$

$$V_2 = \left\{ f : \mathbb{N} \to \mathbb{C} \text{ such that } \sum_{n \geq 0} f(n)x^n = \sum_{i=1}^{k}\sum_{j=1}^{d_i} \frac{\beta_{ij}}{(1-\gamma_i x)^j} \right\}$$

1. $V_2 \subseteq V_1$
2. $\{x^i/Q(x) \mid 0 \leq i < n\}$ spans V_1
3. $\{1/(1-\gamma_i x)^j) \mid 1 \leq i \leq k,\ 1 \leq j \leq d_i\}$ is linearly independent in V_2

□

Fibonacci Numbers

Example

$f(0) = 0,\ f(1) = 1,\ f(n) - f(n-1) - f(n-2) = 0$ for $n \geq 2$

$$\sum_{n\geq 0} f(n)x^n = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 \ldots = \frac{x}{1 - x - x^2}$$

$$= \frac{x}{(1 - \left(\frac{1+\sqrt{5}}{2}\right)x)(1 - \left(\frac{1-\sqrt{5}}{2}\right)x)} = \frac{1/\sqrt{5}}{1 - \left(\frac{1+\sqrt{5}}{2}\right)x} + \frac{-1/\sqrt{5}}{1 - \left(\frac{1-\sqrt{5}}{2}\right)x}$$

Thus,

$$f(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Thank You

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